Theorems V and VI hold word for word when the ternary continued fraction has a finite number of non-periodic partial quotients.

Some progress has been made in the problem of finding a periodic ternary continued fraction which shall be the development of a given cubic irrationality, but the results are not yet in final form.

[1] Lagrange, *Mem. Berlin*, **24**, 1770 *Oev.* II, **74.**

[2] Jacobi, *Ges. Werke*, VI, 385–426.

[3] Bachmann, *Crelle*, **75**, 1873, (25–34).

[4] Berwick, *Proc. London Math. Soc.*, (Ser. 2), **12**, 1913.

A CHARACTERIZATION OF JORDAN REGIONS BY PROPERTIES HAVING NO REFERENCE TO THEIR BOUNDARIES

By Robert L. Moore

Department of Mathematics, University of Pennsylvania

Communicated by E. H. Moore, September 18, 1918

Schoenflies[1] has formulated a set of conditions under which the common boundary of two domains will be a simple closed curve. A different set has been given by J. R. Kline.[2] Carathéodory[3] has obtained conditions under which the boundary of a single domain will be such a curve. In each of these treatments, however, conditions are imposed 1) on the boundary itself, 2) regarding the relation of the boundary to the domain or domains in question. In the present paper I propose to establish the following theorem in which *all the conditions imposed are on the domain R alone.*

Theorem. *In order that a simply*[4] *connected, limited, two-dimensional domain R should have a simple closed curve as its boundary it is necessary and sufficient that R should be uniformly connected im kleinen.*[5]

Proof. Suppose the simply connected domain R is connected im kleinen. Let M denote the boundary of R, that is to say the set of all those limit points of R that do not belong to R.

I will first show that M can not contain two arcs that have in common only one point, that point being an endpoint of only one of them. Suppose it does contain two such arcs EFG and FK with no point in common except F. Let α and β denote circles with common center at F, and with radii r_1 and r_2, respectively, such that $r_1 > r_2$ and such that E, K and G lie without α. By hypothesis there exists a positive number δ_{r_2} such that if X and Y are two points of R at a distance apart less than δ_{r_2} then X and Y lie together in a connected subset of R that lies wholly within some circle of radius r_2. Let γ denote a circle with center at F and with a radius less than one half of δ_{r_2} and also less than $r_1 - r_2$. It is clear that if two points of R are both within γ then they lie together in a connected subset of R that lies wholly within α. There

exist (fig. 1), within γ, points E', G' and K' and arcs $E'K'$ and $K'G'$ such that 1) E' and G' are on the arc EFG in the order $EE'FG'G$ and K' is on the arc FK, 2) the interval $E'G'$ of the arc EFG lies wholly within γ, 3) $E'K'$ has in common with EFG only the point E' and, in common with FK, only the point K', 4) $K'G'$ has in common with EFG only the point G' and, in common with FK, only the point K', 5) the segment FK' of the arc FK lies wholly within the closed curve J formed by the arcs $E'K'$ and $K'G'$ and the interval $E'FG'$ of the arc EFG. The arc $E'K'$ together with the interval $E'F$ of EFG and the interval FK' of FK form a closed curve J_1. The arc $K'G'$ together with the interval FG' of EFG and the interval FK' of FK form a closed curve J_2. Let I, I_1 and I_2 denote the interiors of J, J_1 and J_2 respectively. Then I is the sum of I_1, I_2 and the segment FK' of the arc FK. Let P, P_1 and P_2 denote points on the segments FK', $E'F$ and FG' respectively. The point P is a

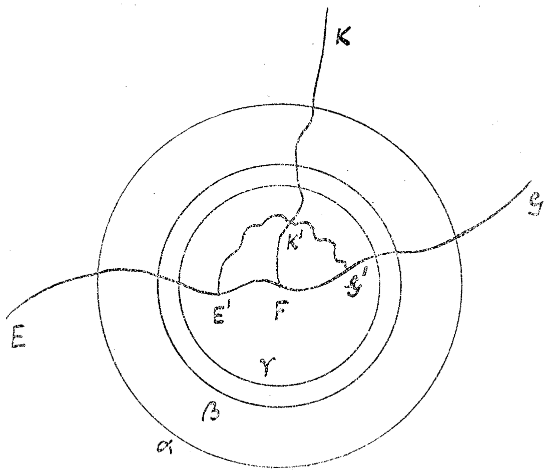

FIG. 1

boundary point of R and the region I contains P. Hence there must be points of R in I and therefore in I_1 or in I_2. The point P_1 also is on the boundary of R. It follows that there exist, within the circle γ, points of R lying either without J (and therefore without both J_1 and J_2) or within J_1. Similarly either I_2 or the exterior of J contains points of R that are within γ. Thus either both I_1 and I_2 contain points of R or only one of them contains points of R in which latter cases there must exist points of R that are within γ and without J. But if a connected point-set lies entirely within α and contains either 1) a point of I_1 and a point of I_2 or 2) a point of I_1 or I_2 and a point without J; then it must clearly contain at least one point of EFG or of FK. Therefore it can not be a subset of R. Thus the supposition that M contains two arcs EFG and FK with only the point F in common leads to a contradiction.

That M, the boundary of R, is connected is a consequence of a theorem of Hausdorff's.[6] I will proceed to show that it is connected im kleinen. Sup-

pose that it is not. Then there exists an infinite sequence $P, P_1, P_2, P_3, \ldots$ of distinct points belonging to M and a sequence of positive numbers e, e_1, $e_2, e_3, \ldots$ such that 1) $e/2 > e_1 > e_2 > e_3 > \ldots$, 2) $\mathrm{Lim.}_{n=\infty}\, e_n = 0$, 3) the distance from P_n to P is equal to e_n, 4) if n is a positive integer there exists no connected sub-set of M containing P_n and P such that all the points of this sub-set are at a distance of less than e from P. Let K denote a circle with center at P and radius $e/2$. The point-set M contains a closed connected subset t_n that contains P_n and at least one point on K but no point without K. There does not exist an infinite sequence of distinct positive integers $\bar{n}_1, \bar{n}_2, \bar{n}_3, \ldots$ such that, for every m, $t_{\bar{n}_m}$ has a point in common with t_1. For if such were the case then $P + t_1 + t_{\bar{n}_1} + t_{\bar{n}_2} + \ldots$ would be a connected point-set containing P_1 and P and lying entirely within a circle with center at P and radius e. It follows that there exists a positive integer n_1 greater than 1 such that if $n \geqq n_1$ then t_n has no point in common with t_1. Similarly there exists n_2 greater than n_1 such that if $n \geqq n_2$ then t_n has no point in common with t_{n_1}. This process may be continued. It follows that M contains an infinite sequence of closed connected point-sets $t_{n_1}, t_{n_2}, t_{n_3}, \ldots$ $(n_1 < n_2 < n_3 < \ldots)$ such that no two of them have a point in common and such that, for every m, t_{n_m} contains P_{n_m} and at least one point on K but no point without K. Let $\overline{K}$ denote the circle whose center is P and whose radius is e_1. For every m, t_{n_m} contains a closed, connected subset $\bar{t}_m$ such that 1) every point of $\bar{t}_m$ is either on K or $\overline{K}$ or between K and $\overline{K}$, 2) $\bar{t}_m$ contains at least one point on K and at least one point on $\overline{K}$. Let K^* denote a circle concentric with K and $\overline{K}$ and lying between them. There exists on K^* a point O and a sequence of points $O_1, O_2, O_3, \ldots$ such that O_m belongs to $\bar{t}_m$ and such that O is a limit point of the point-set $O_1 + O_2 + O_3 + \ldots$. Let K' denote a circle lying between K and $\overline{K}$ and with center at O. Let $2h$ denote its radius. Let K'' denote a circle with the same center and with a radius less than h and also less than $\delta_h/2$. There exist within K'' four points $O_{m_1}, O_{m_2}, O_{m_3}$ and O_{m_4} of the set $O_1, O_2, O_3, \ldots$. Of these four points, two, A_1 and A_3, are separated from each other on K^* by the other two, A_2 and A_4. For each i $(1 \leqq i \leqq 4)$ let a_i denote that one of the point-sets $t_{m_1}, t_{m_2}, t_{m_3}$ and t_{m_4} to which the point A_i belongs. For each i $(1 \leqq \mathrm{i} \leqq 4)$ there exists, within K'' (fig. 2), a circle K_i, with center at A_i, that neither encloses nor contains a point of any of the sets a_1, a_2, a_3, a_4, except a_i. The circles K_1 and K_3 respectively enclose points B_1 and B_3 belonging to R. Every closed connected point-set that contains both B_1 and B_3 and lies between K and $\overline{K}$ must contain a point either of a_2 or of a_4. But there exists a connected subset of R that contains both B_1 and B_3 and lies wholly with K'. Thus the supposition that M is not connected im kleinen has led to a contradiction.

Since M is limited, closed, connected and connected im kleinen, it follows[7] that every two points of M can be joined by a simple continuous arc lying wholly in M. The point-set M contains a countable subset N of points

$P_1, P_2, P_3, \ldots$ such that every point of M is a limit point of N. For each n, P_n can be joined to P_{n+1} by a simple continuous arc P_nP_{n+1} lying entirely in M. In view of the fact that if M contains two arcs with only one point in common, that point must be an end-point of each of them, it is clear that, for every n, $P_1P_2 + P_2P_3 + \ldots + P_{n-1n}P$ is either a simple closed curve or a simple continuous arc. It is clear that if, for some n, it is a simple closed curve, then this curve is identical with M. Suppose on the other hand, that, for every n, it is a simple continuous arc A_nB_n, the notation being so assigned that, in the order from A_n to B_n on the arc A_nB_n, every A_i precedes every B_j $(1 \leqq i \leqq n, 1 \leqq j \leqq n)$, Let N^* denote the point-set constituted by the sum of all the arcs $A_1B_1, A_2B_2, A_3B_3, \ldots$ If $P_1, P_2, P_3, \ldots$ is a

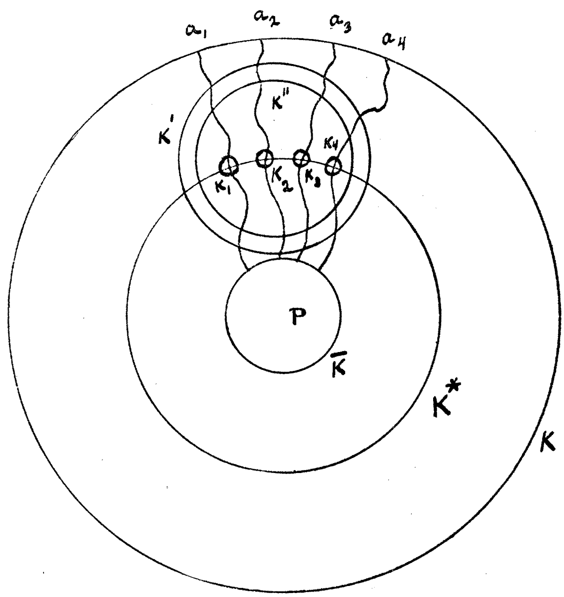

FIG.

set of points such that for every n, P_n follows B_n on some arc of the set A_1B_1, $A_2B_2, \ldots$ then the set $P_1, P_2, P_3, \ldots$ does not have more than one limit point. For suppose there exists such a set with two distinct limit points O_1 and O_2. Then if C_1 and C_2 are two distinct circles with center at O_1 and such that O_2 is without each of them, there clearly exists an infinite set of arcs G (Fig. 3) such that 1) each arc of G is, for some n, i and j $(1 \leqq i \leqq n, 1 \leqq j \leqq n)$ a sub-interval of the interval B_iB_j of the arc A_nB_n, 2) no two arcs of G have a point in common, 3) each arc of G lies entirely between C_1 and C_2 except that one of its endpoints is on C_1 and the other one is on C_2. But, in the course of the above proof that M is connected im kleinen, it was shown that M does not contain such a set of arcs.

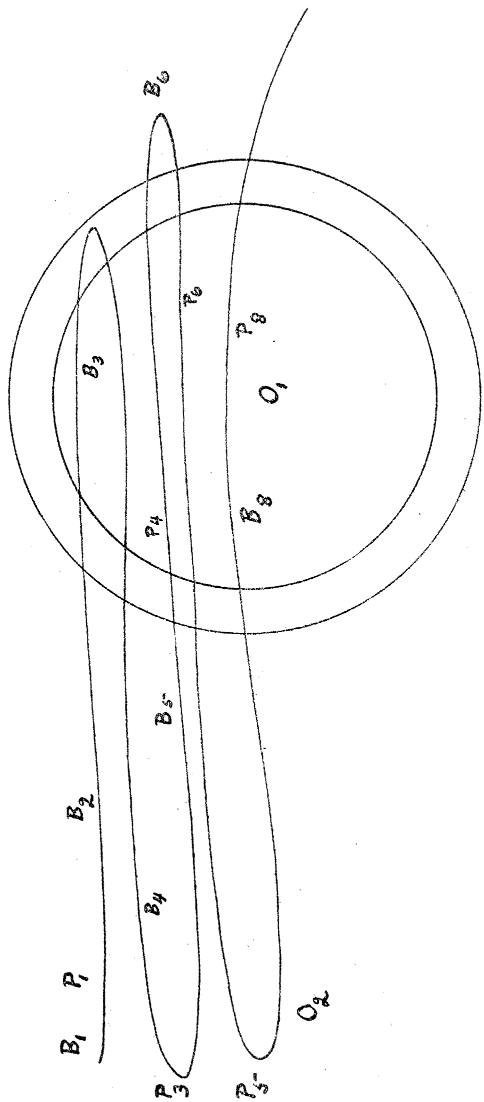

FIG. 3

Similarly if P_1, P_2, P_3, . . . is a set of points such that, for every n, P_n precedes A_n on some arc of the set A_1B_1, A_2B_2, . . . , then the set P_1, P_2, P_3, . . . can not have more than one limit point. It is clear that at least one of the sets A_1, A_2, A_3, . . . , B_1, B_2, B_3, . . . is infinite. Otherwise N^*, and therefore M, would be a simple continuous arc, which is impossible in view of the fact that it is the boundary of a limited domain. There remain two conceivable cases. Either 1) the sets A_1, A_2, A_3, . . . and B_1, B_2, B_3, . . . are both infinite or 2) one is finite and the other infinite. Suppose they are both infinite. Consider the facts that 1) no set with the properties indicated above for P_1, P_2, P_3, . . . has more than one limit point, 2) no simple continuous arc is the boundary of a limited domain and, 3) M does not contain two arcs with only one common point, that point being an end-point of only one of them. In view of these facts it is clear that A_1, A_2, A_3, . . . and B_1, B_2, B_3, . . . have a common limit point O and that the point-set $N^* + O$ is a simple closed curve, identical with M.

Suppose the set A_1, A_2, A_3, . . . is infinite while B_1, B_2, B_3, . . . is not. With the aid of the same three considerations, it is clear that in this case A_1, A_2, A_3, . . . has as its only limit point the point B_m where m is a positive integer such that, for every n greater than m, B_n coincides with B_m. In this case the pount-set N^* is a simple closed curve, identical with M. Of course a similar argument applies in case A_1, A_2, A_3, . . . is finite and B_1, B_2, B_3, . . . is infinite. Thus in every case M is a simple closed curve.

I will now proceed to establish the converse proposition that *every Jordan region is uniformly connected im kleinen.* Suppose that, on the contrary, there exists a closed curve J whose interior I does not have this property. Then there must evidently exist a positive number a, a point O on J and two infinite sequences of points X_1, X_2, X_3, . . . Y_2, Y_2, Y_3, . . . lying in I, such that O is the sequential limit point of each of these sequences and such that for no n can X_n be joined to Y_n by a connected subset of I that lies entirely within a circle K with center at O and radius a. But there exists[8] a closed curve J^* containing O such that every point of J^* belongs either to J or to K and such that every point within J^* is within both J and K. It is clear that for some n the points X_n and Y_n are both within J^*. But the interior of J^* is a connected subset of I and also of the interior of K. Thus the supposition that I is not uniformly connected im kleinen has led to a contradiction.

[1] Schoenflies, A., *Göttingen, Nachr. Ges. Wiss.*, 1902, (185).

[2] Kline, J. R., *Bull. Amer. Math. Soc., New York*, **24**, 1918, (471). One of Kline's conditions is that the common boundary of the domains in question should be connected 'im kleinen.'

[3] Carathéodory, C., *Math. Ann., Leipzig*, **73**, 1912–13, (366).

[4] In space of two dimensions, a domain is said to be *simply connected* if it is connected and contains every limited point-set whose boundary lies in it.

[5] A point-set M is said to be *connected 'im kleinen'* if for every point P of M and every positive number e there exists a positive number δ_{ep} such that if X is a point of M at a dis-

tance less than δ_{ep} from the point P, then X and P lie together in a connected sub-set of M every point of which is at a distance of less than e from the point P. The set M is said to be *uniformly* connected im kleinen if for every positive number e there exists a positive number δ_e such that if P_1 and P_2 are two points of M at a distance apart less than δ_e then they lie together in a connected sub-set of M every point of which is at a distance of less than e from P_1. Cf. Hahn, H., *Jahresber. D. Math. Ver., Leipzig,* **23**, 1914, (318–322).

[6] Hausdorff, F., *Grundzüge der Mengenlehre,* Veit & Co., Leipzig, 1914.

[7] Cf. my paper, A theorem concerning continuous curves, *Bull. Amer. Math. Soc.,* **23**, 1917, (233–236).

[8] Cf. Theorem 43 of my paper, On the foundations of plane analysis situs, *Trans. Amer. Math. Soc., New York,* **17**, 1916, (131–164). I take this opportunity to correct an error in the statement of Theorem 44 of this paper. In this statement the upper 'interior' and the two upper 'without's are to be omitted.

A BIOMETRIC STUDY OF HUMAN BASAL METABOLISM

By J. Arthur Harris and Francis G. Benedict

Nutrition Laboratory and Station for Experimental Evolution, Carnegie Institution of Washington

Communicated October 8, 1918

Investigators are now generally agreed that the metabolism, expressed in terms of calories per unit of time, of the normal subject shall be taken as a basis of comparison in the investigation of all the special problems of human nutrition, for example, that of the requirements for muscular activity, that of the influence of specific diseases or of the level of nutrition upon metabolism, that of the change of metabolic activity with age, and so forth. Critical investigations in both European and American laboratories have shown that the gaseous metabolism is so affected by various factors that determinations which are to serve as a standard must be made under very exactly controlled conditions. It is not merely necessary to devise apparatus in which the physical difficulties of direct calorimetry (or of the exact measurement of gaseous exchange from which heat production may be computed) are overcome. Certain biological factors must be ruled out. Those of greatest importance as sources of experimental error are muscular activity and the stimulatory action of recently ingested food. The heat production of the individual in a state of complete muscular repose 12–14 hours after the last meal, i.e., in the postabsorptive condition, has been called the basal metabolism.

For a decade the Nutrition Laboratory has been engaged in carrying out a series of determinations of basal metabolism in normal human individuals of both sexes and of widely different ages. These have been made with all the modern refinements of method and manipulation. The subjects were in presumably good health. All those with febrile temperature were discarded. All were in the postabsorptive condition. Perfect muscular repose during the short periods required for indirect calorimetry was assured by an automatic record of all movements, even those imperceptible to a trained observer.